The Second Hankel Determinant for Starlike Functions of Order Alpha

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Abstract

Let f be analytic in $D = \{z : |z| < 1\}$ with $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. We give sharp bounds for the second Hankel determinant $H_2(2) = |a_2a_4 - a_3^2|$ when f is starlike of order α.

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Introduction, definitions and preliminaries

Let S be the class of analytic normalised univalent functions f, defined in $z \in D = \{z : |z| < 1\}$ and given by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

The qth Hankel determinant of f is defined for $q \geq 1$ and $n \geq 0$ as follows, and has been extensively studied, see e.g. [1, 4, 5].

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} \cdots & a_{n+q+1} \\ a_{n+1} & \cdots & \vdots \\ \vdots & & \\ a_{n+q-1} & \cdots & a_{n+2q-2} \end{vmatrix}.$$

Denote by S^* the subclass of S of starlike functions, so that $f \in S^*$ if, and only if, for $z \in D$

$$Re\ \frac{zf'(z)}{f(z)} > 0.$$

Suppose that f is analytic in D and given by (1). Then f is starlike of order α in D if, and only if, for $0 \leq \alpha < 1$,

$$Re\ \frac{zf'(z)}{f(z)} > \alpha. \tag{2}$$

We denote this class by $S(\alpha)$, so that $S(0) = S^* \subset S$ and $S^*(\alpha) \subset S^*$.

Let P be the class of functions p satisfying $Re\ p(z) > 0$ for $z \in D$, with $p(0) = 1$.

Write

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n. \tag{3}$$

We shall need the following result [3], which has been used widely.

Lemma

Let $p \in P$ and be given by (3), then for some complex valued y with $|y| \leq 1$, and some complex valued ζ with $|\zeta| \leq 1$

$$2p_2 = p_1^2 + y(4 - p_1^2),$$
$$4p_3 = p_1^3 + 2(4 - p_1^2)p_1 y - p_1(4 - p_1^2)y^2 + 2(4 - p_1^2)(1 - |y|^2)\zeta.$$

Results

It was shown in [2] that if $f \in S^*$ then $H_2(2) \leq 1$, and in [6] that if $f \in S(\alpha)$ with $0 \leq \alpha \leq \frac{1}{2}$, then $H_2(2) \leq (1 - \alpha)^2$, both inequalities are sharp.

We give the complete solution for $f \in S(\alpha)$ when $0 \leq \alpha < 1$ as follows.

Theorem

If $f \in S^*(\alpha)$, then for $0 \leq \alpha < 1$, the second Hankel determinant

$$H_2(2) \leq (1-\alpha)^2.$$

The inequality is sharp.

Proof. It follows from (2) that we can write $zf'(z) = \alpha + (1-\alpha)f(z)p(z)$, and so equating coefficients we obtain

$$\begin{aligned}
a_2 &= (1-\alpha)p_1 \\
a_3 &= \frac{1}{4}(2(1-\alpha)^2p_1^2 + 2p_2 - 2\alpha p_2) \\
a_4 &= \frac{1}{6}(1-\alpha)((1-\alpha)^2p_1^3 + 3(1-\alpha)p_1p_2 + 2p_3).
\end{aligned}$$

Hence

$$\begin{aligned}
|a_2a_4 - a_3^2| = |&-\frac{1}{48}(1-\alpha)^2(3-8\alpha+4\alpha^2)p_1^4 \\
&-\frac{1}{4}p_2^2 + \frac{1}{2}\alpha p_2^2 - \frac{1}{4}\alpha^2p_2^2 + \frac{1}{3}p_1p_3 - \frac{2}{3}\alpha p_1p_3 + \frac{1}{3}\alpha^2p_1p_3|.
\end{aligned}$$

Noting that without loss in generality we can write $p_1 = p$, with $0 \leq p \leq 2$, we now use the Lemma to express the above in terms of p to obtain

$$\begin{aligned}
|a_2a_4 - a_3^2| = |&-\frac{1}{48}(1-\alpha)^2(3-8\alpha+4\alpha^2)p^4 \\
&+\frac{1}{24}(1-\alpha)^2p^2(4-p^2)y - \frac{1}{12}(1-\alpha)^2p^2(4-p^2)y^2 \\
&-\frac{1}{16}(1-\alpha)^2(4-p^2)^2y^2 + \frac{1}{6}(1-\alpha)^2p(4-p^2)(1-|y|^2)\zeta| \\
&\leq \frac{1}{48}(1-\alpha)^2|(3-8\alpha+4\alpha^2)|p^4 \\
&+\frac{1}{24}(1-\alpha)^2p^2(4-p^2)|y| + \frac{1}{12}(1-\alpha)^2p^2(4-p^2)|y|^2 \\
&+\frac{1}{16}(1-\alpha)^2(4-p^2)^2|y|^2 + \frac{1}{6}(1-\alpha)^2p(4-p^2)(1-|y|^2) := \phi(|y|).
\end{aligned}$$

It is a simple exercise to show that $\phi'(|y|) \geq 0$ on $[0,\ 1]$, so $\phi(|y|) \leq \phi(1)$. Putting $|y| = 1$ gives

$$
\begin{aligned}
|a_2a_4 - a_3^2| \leq & \frac{1}{48}(1-\alpha)^2|(3-8\alpha+4\alpha^2)|p^4 \\
& + \frac{1}{8}(1-\alpha)^2p^2(4-p^2) + \frac{1}{16}(1-\alpha)^2(4-p^2)^2 \\
& = 1 - 2\alpha + \alpha^2 - \frac{1}{16}(1-\alpha)^2p^4 + \frac{1}{48}(1-\alpha)^2p^4|3-8\alpha+4\alpha^2|.
\end{aligned}
$$

Considering $3-8\alpha+4\alpha^2 \geq 0$ and $3-8\alpha+4\alpha^2 \leq 0$ separately, elementary calculus shows that the above expression is bounded by $(1-\alpha)^2$ in both cases.

We note that the inequality in the Theorem is sharp when $p_1 = p_3 = 0$ and $p_2 = 2$.

$\square$

References

[1] W.K. Hayman, On the second Hankel determinant of mean univalent functions, *Proc, Lond. Math. Soc.*. **3** no. 18 (1968) 77–94.

[2] A. Janteng, S. Halim, & M. Darus, Hankel Determinants for Starlike and Convex Functions, *Int. Journal. Math. Analysis.* **1** no. 13 (2007) 619–625.

[3] R. J. Libera & E. J. Zlotkiewicz, Coefficient bounds for the inverse of a function with derivative in P. *Proc. Amer. Math. Soc.* **87** no. 2 (1983), 251–257.

[4] J. W. Noonan & D. K. Thomas, On the second Hankel determinant of areally mean p-valent functions, *Trans. Amer. Math. Soc.*, **223** (2)(1976): 337-346.

[5] Ch. Pommerenke, On the Hankel determinants of univalent functions, Mathematika (London) **16** no. 13 (1967) 108–112.

[6] D. Vamshee Krishna & T. Ramreddy, Coefficient Inequality for Certain Subclasses of Analytic Functions, *New Zealand Journal of Mathematics*, **42** (2012) 217–228.